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LETTER TO THE EDITOR

Generalized deformed SU(2) algebra

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Abstract. A generalized deformed algebra $SU_\Phi(2)$, characterized by a structure function Φ, is obtained. The usual SU(2) and $SU_q(2)$ algebras correspond to specific choices of the structure function Φ. The action of the generators of the algebra on the relevant basis vectors, as well as the eigenvalues of the Casimir operator, are easily obtained. Possible applications in improving phenomenological models are discussed.

Quantum algebras [1–4] (also called quantum groups) are nonlinear deformations of the corresponding Lie algebras, to which they reduce when the deformation parameter q is set equal to one. They have recently found several applications in physics, especially after the introduction of the q-deformed harmonic oscillator [5, 6]. The $SU_q(2)$ symmetry, in particular, has been widely used for the description of rotational spectra of deformed nuclei [7–9], superdeformed nuclei [10], and diatomic molecules [11–14], as well as for the description of the electromagnetic transition probabilities connecting these levels [15]. The introduction of generalized deformed oscillators [16] has led to more applications in physics, since it gives the possibility of constructing oscillators behaving like a physical system, the Morse oscillator [17] for example. Although several generalized deformed oscillators, as well as unification schemes for them, have been introduced (see [18] for a list of references), attempts at generalizing the $SU_q(2)$ symmetry only recently began appearing [19–21]. A generalized version of the deformed SU(2) can be useful in improving [9] the agreement of phenomenological models to experimental data.

In the present work we construct a generalized deformed SU(2) algebra, characterized by a structure function Φ. The usual SU(2) and $SU_q(2)$ algebras are obtained for specific forms of the structure function, but additional forms are possible. The present method allows for the determination of the action of the generators on the basis vectors and of the eigenvalues of the Casimir operator in a simple way. Its possible usefulness in physical applications is also discussed.

We start the construction of the algebra in a very general way, adding the necessary restrictions as we proceed. Consider a Hilbert space V, consisting of the tensor sum of the subspaces V_l, i.e.

$$V = \oplus_{l=0}^{\infty} V_l \tag{1}$$

where the subspaces V_l are unitary subspaces of dimension $2l+1$ and basis vectors $|l, m\rangle$ with l integer or half-integer (in what follows we will denote the set of integers and half-integers by I_l) and m taking values from the set $S_l = \{-l, -l+1, \ldots, l-1, l\}$. The basis vectors are orthonormal

$$\langle l', m' | l, m\rangle = \delta_{ll'}\delta_{mm'}$$

and cover V

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |l, m\rangle\langle l, m| = 1.$$

In V we consider the operators J_0, J_+, J_-, the action of which on the basis vectors is given by

$$J_0|l, m\rangle = m|l, m\rangle \qquad m \in S_l \qquad l \in I_l \tag{2}$$

$$J_+|l, m\rangle = A(l, m)|l, m+1\rangle \qquad m \in S_l \qquad l \in I_l \tag{3}$$

$$J_+|l, l\rangle = 0 \tag{4}$$

$$J_- = (J_+)^\dagger \tag{5}$$

where $A(l, m)$ is a real entire function defined for $m \in [-l, l]$, $l \in [0, \infty)$, satisfying the equations

$$A(l, l) = 0 \tag{6}$$

$$A(l, -l-1) = 0. \tag{7}$$

It is clear that of interest are the values of $A(l, m)$ with $l \in I_l$, $m \in S_l - \{l\}$.

From equations (2)–(5) we immediately obtain

$$J_0 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} m|l, m\rangle\langle l, m| \tag{8}$$

$$J_+ = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A(l, m)|l, m+1\rangle\langle l, m| \tag{9}$$

$$J_- = \sum_{l=0}^{\infty}\sum_{m=-1}^{l} A(l, m)|l, m\rangle\langle l, m+1|. \tag{10}$$

Using equations (8)–(10) one can easily prove that

$$[J_0, J_+] = J_+ \tag{11}$$

$$[J_0, J_-] = -J_- \tag{12}$$

$$J_0^n J_+ = J_+(J_0+1)^n \tag{13}$$

$$J_0^n J_- = J_-(J_0-1)^n. \tag{14}$$

Then for every entire function A one has

$$A(J_0)J_+ = J_+A(J_0+1) \tag{15}$$

$$A(J_0)J_- = J_-A(J_0-1). \tag{16}$$

From (9) and (10) one easily obtains

$$J_+J_- = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A(l, m-1))^2 |l, m\rangle\langle l, m| \tag{17}$$

$$J_-J_+ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (A(l, m))^2 |l, m\rangle\langle l, m|. \tag{18}$$

One can define an operator J such that

$$J|l, m\rangle = l|l, m\rangle \qquad m \in S_l \qquad l \in I_l. \tag{19}$$

Clearly one has

$$J = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} l|l, m\rangle\langle l, m| \tag{20}$$

and

$$(A(J, J_0))^2|l, m\rangle = (A(l, m))^2|l, m\rangle. \tag{21}$$

Then (17) and (18) can be written as

$$J_+J_- = (A(J, J_0 - 1))^2 \tag{22}$$

$$J_-J_+ = (A(J, J_0))^2. \tag{23}$$

For the commutator of J_+ with J_- one has from (17) and (18) that

$$[J_+, J_-] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} ((A(l, m-1))^2 - (A(l, m))^2)|l, m\rangle\langle l, m| \tag{24}$$

while from (22) and (23) one finds

$$[J_+ \cdot J_-] = (A(J, J_0 - 1))^2 - (A(J, J_0))^2. \tag{25}$$

In what follows we wish to restrict ourselves to operators J_0, J_+, J_- which close an algebra by themselves, i.e. without involving J. Equations (11) and (12) already do not involve J, but (25) does. We wish to restrict ourselves to algebras for which the right-hand side (RHS) of (25) is a function of J_0 only. We assume that this function of J_0 can be written in the form $B(J_0) - B(J_0 - 1)$, i.e. we require that

$$[J_+, J_-] = B(J_0) - B(J_0 - 1). \tag{26}$$

(For sufficient conditions under which a function of J_0 can be written in the form $B(J_0) - B(J_0 - 1)$ see [22].) By equating the RHS of (25) and (26) and acting on the basis vector $|l, m\rangle$ we find that for every $m \in S_l$ and $l \in I_l$ the following condition should be satisfied

$$(A(l, m))^2 - (A(l, m-1))^2 = B(m-1) - B(m). \tag{27}$$

This condition is satisfied if $(A(l,m))^2$ is separable into the difference of a function of l and a function of m, i.e.

$$(A(l,m))^2 = C(l) - B(m) \qquad m \in S_l \qquad l \in I_l. \tag{28}$$

From (6), (7) and (28) one easily sees that

$$C(l) = B(l) \tag{29}$$

and

$$B(l) = B(-l-1). \tag{30}$$

The last equation implies that $B(l)$ is of the form

$$B(l) = \Phi(l(l+1)). \tag{31}$$

This can be proven as follows: in (30) one can put $l = j - \frac{1}{2}$. Then $B(j-\frac{1}{2}) = B(-j-\frac{1}{2})$. Thus the function $G(j) = B(j-\frac{1}{2})$ is an even function of j, i.e. $G(j) = G(-j)$. For every even function $G(j)$ one can find a function F such that $G(j) = F(j^2)$, which implies that $B(l) = F(l^2 + l + \frac{1}{4})$. As a result there is a function $\Phi(x) = F(x + \frac{1}{4})$, for which (31) is valid. (The inverse also holds: for every function of the form given in (31), equation (30) is satisfied, as one can trivially see.)

From (28), (29) and (31) one then finds that

$$(A(l,m))^2 = \Phi(l(l+1)) - \Phi(m(m+1)) \qquad m \in S_l \qquad l \in I_l. \tag{32}$$

This can be written as

$$(A(J,J_0))^2 = \Phi(J(J+1)) - \Phi(J_0(J_0+1)) \tag{33}$$

because of the following general proposition:

$$F(l,m) = 0 \qquad m \in S_l, l \in I_l \Leftrightarrow F(J,J_0) = 0 \tag{34}$$

where $F(l,m)$ is any entire function. The proof of the proposition is simple. From $F(l,m) = 0$ one has $F(l,m)|l,m\rangle = 0$ and then $F(J,J_0)|l,m\rangle = 0$ for every $m \in S_l$ and $l \in I_l$, which implies that $F(J,J_0) = 0$. The inverse is also proved through the same steps.

From (33) it is clear that $\Phi(x)$ must be an increasing function for $x > 0$. Thus the restricted as described above algebra satisfies the relations

$$[J_0, J_+] = J_+ \qquad [J_0, J_-] = -J_- \tag{35}$$

$$J_- J_+ = \Phi(J(J+1)) - \Phi(J_0(J_0+1)) \tag{36}$$

$$J_+ J_- = \Phi(J(J+1)) - \Phi(J_0(J_0-1)) \tag{37}$$

$$[J_+, J_-] = \Phi(J_0(J_0+1)) - \Phi(J_0(J_0-1)) \tag{38}$$

where $\Phi(x)$ is any increasing entire function defined for $x \geqslant -\frac{1}{4}$. This algebra is a generalization of SU(2), characterized by the function Φ. Therefore we are going to use for this the symbol $SU_\Phi(2)$.

Using (35)–(38) one can easily verify that the Casimir operator (which commutes with all the generators of the algebra) is

$$C = J_- J_+ + \Phi(J_0(J_0+1)) = J_+ J_- + \Phi(J_0(J_0-1)). \tag{39}$$

From (36) one then has

$$C = \Phi(J(J+1)). \tag{40}$$

From this equation it is clear that the eigenvalues of the Casimir operator in the basis $|l, m\rangle$ are $\Phi(l(l+1))$, with $l=0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The action of the various operators on the basis vectors is summarized by

$$J_0|l, m\rangle = m|l, m\rangle \tag{41}$$

$$J_+|l, m\rangle = \sqrt{\Phi(l(l+1)) - \Phi(m(m+1))}\,|l, m+1\rangle \tag{42}$$

$$J_-|l, m\rangle = \sqrt{\Phi(l(l+1)) - \Phi(m(m-1))}\,|l, m-1\rangle \tag{43}$$

$$C|l, m\rangle = \Phi(l(l+1))|l, m\rangle. \tag{44}$$

We have therefore constructed an algebra $SU_\Phi(2)$, which is a generalization of the SU(2) algebra characterized by the *structure function* Φ. A few comments are now in place:

(i) The usual SU(2) algebra is obtained for

$$\Phi(x(x+1)) = x(x+1)$$

as one can see from equations (35)–(44).

(ii) The quantum algebra $SU_q(2)$, with commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0]_q \tag{45}$$

is obtained for

$$\Phi(x(x+1)) = [x]_q[x+1]_q \tag{46}$$

with q-numbers defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

One can be persuaded that the function $\Phi(x(x+1))$ given in (46) is really a function of the variable $x(x+1)$ (a fact that is not immediately obvious) by having a look at the Taylor expansions given in [8] (10*a*) and (10*b*).

(iii) In [19] the following formalism is used:

$$[J_+, J_-] = f(J_0) \qquad C = J_-J_+ + h(J_0) \tag{47}$$

and the condition

$$f(J_0) = h(J_0) - h(J_0 - 1) \tag{48}$$

is found to hold. Similar formalisms have been used in [20, 21]. These results correspond to

$$h(J_0) = \Phi(J_0(J_0+1)) \qquad f(J_0) = \Phi(J_0(J_0+1)) - \Phi(J_0(J_0-1))$$

which automatically satisfy the condition (48). In the present method the extra results of (42)–(44) are obtained at no toil.

(iv) It is clear that the RHS of (38) is an odd function of J_0. This imposes an extra restriction on $f(J_0)$ of the previous formalism (equation (47)), while it is automatically satisfied in the case of $SU_q(2)$, as one can easily see in (45).

(v) In [9] it has been argued that the Hamiltonian

$$E(J) = a[\sqrt{1 + bJ(J+1)} - 1] \tag{49}$$

gives better agreement to rotational nuclear spectra than the one coming from the $SU_q(2)$ symmetry [7, 8]. Using the present technique one can construct an $SU_\Phi(2)$

algebra giving the spectrum of (49) exactly. This algebra is characterized by the structure function

$$\Phi(J(J+1)) = a[\sqrt{1+bJ(J+1)} - 1].$$

It is of interest to check if this choice of structure function also improves the agreement between theory and experiment in the case of the electromagnetic transition probabilities connecting these energy levels. In order to study this problem, one has to construct the relevant generalized Clebsch–Gordan coefficients [15]. Work in this direction is in progress.

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